K22P 3320

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, April 2022
(2018 Admission Onwards)
MATHEMATICS
MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks.

1. Sketch the level set and graph of the function $f(x_1, x_2) = x_1 - x_2$.

2. Show that the set S of all unit vectors at all points of $\mathbb{R}^2$ form a 3-surface in $\mathbb{R}^4$.

3. Prove that a parametrized curve $\alpha : I \to S$ is a geodesic in S if and only if its covariant acceleration $[\dot{\alpha}]'$ is zero along α.

4. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$ and let α be a parametrized curve in S from p to q. Then prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vectorspace isomorphism.

5. Show that the length of a parametrized curve is invariant under re-parametrization.

6. Express torus as a parametrized surface in $\mathbb{R}^4$.

PART – B

Answer **any 4** questions without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Sketch the vector field $\mathbb{X}(p) = (p, X(p))$ on $\mathbb{R}^2$, where $X(x_1, x_2) = (x_2, x_1)$. Also find the integral curve through an arbitrary point (a, b).

   b) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\Delta f(p)]^{\perp}$.

P.T.O.

8. a) State and prove Lagrange multiplier theorem in an n-surface in $\mathbb{R}^{n+1}$.

b) If $S \subset \mathbb{R}^{n+1}$ is a connected n-surface in $\mathbb{R}^{n+1}$, then prove that on S there exist only two orientations.

c) Discuss about the orientability of Mobious band.

9. a) Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exist a unique maximal integral curve α of $\mathbb{X}$ with $\alpha(0) = p$ and any other integral curve β with $\beta(0) = p$ will be a restriction of α.

b) Define the special linear group SL(2). Show that it will form a surface.

Unit – II

10. a) Let S be a regular, compact connected oriented n-surface in $\mathbb{R}^{n+1}$, exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Then show that the Gauss map maps S on to the unit sphere S^n.

b) Define a geodesic and show that a geodesic have constant speed.

11. a) Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some $a, b, c, d \in \mathbb{R}$.

b) Define Levi-Civita parallel vector field on a surface S. Also state and prove five properties of the Levi-Civita parallelism.

c) Find the Weingarten map of the cylinder $x_1^2 + x_2^2 = a^2$ of radius $a > 0$ in $\mathbb{R}^3$.

12. a) Show that every oriented plane curve has a local parametrization and the local parametrization of a plane curve is unique up to re-parametrization.

b) Let C be a circle $f^{-1}(r^2)$ where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ oriented by the outward normal $\frac{\nabla f}{||\nabla f||}$. Then obtain a global parametrization of C.

Unit – III

13. a) Let C be an oriented plane curve. Then prove that there exist a global parametrization of C if and only if C is connected.

b) Show that a line integral is invariant under re-parametrization.

14. a) Give an example of a 1-form on $\mathbb{R}^2 - \{0\}$, which is not exact.

b) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let v be a unit vector in S_p, $p \in S$. Then prove that there exist an open set $V \subset \mathbb{R}^{n+1}$ containing p such that $S \cap \mathcal{N}(v) \cap V$ is a plane curve. More over show that the curvature at p of this curve, (suitably oriented) is equal to the normal curvature $\mathcal{K}(v)$.

15. a) On each compact oriented n-surface S in $\mathbb{R}^{n+1}$ prove that there exist a point p such that the second fundamental form at p is definite.

b) Let $\varphi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exist an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$.
